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First Semester Examination 2023/2024 Session

MATH 303: Advanced Real Analysis I

INSTRUCTION: Answer Any FOUR (4) questions **DURATION: 2 HOURS**

1. (a) Define a metric space and show that $(c([a,b]),\rho)$ is a metric space, where $c([a,b])$ is the space of all continuous real valued functions defined on $[a,b]$ and ρ is a function defined on $c([a,b])$ by $\rho(f,g) = \underset{t\in[a,b]}{Max}\{|f(t)-g(t)|\}$ with $f,g \in c([a,b])$.

   (b) Show that (c,ρ) is a complete metric space, where c is the set of all convergent sequence of real numbers and ρ is defined as $\rho(x,y) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$

2. (a) Define an open set in a metric space (X,ρ) and show that every open ball is an open set.

   (b) Show that an arbitrary union of open sets in (X,ρ) is open set in (X,ρ).

3. (a) Define a cluster point of a subset F of a metric space (X,ρ) and show that the arbitrary intersection of closed set is closed.

   (b) Show that $x \in \bar{F}$ if and only if there exists a sequence (x_n) in F such that (x_n) converges to x as $n \to \infty$.

4. (a) Define a complete metric space and show that a subset A of a complete metric space is closed if and only if A is complete.

   (b) If (X,ρ) is a complete metric space and (F_n) is a decreasing sequence of closed sets in (X,ρ) such that diam$(F_n)\to 0$ as $n\to\infty$, then show that $\bigcap F_n$ consist of exactly one point.

5. (a) Define an interior point of a subset A of a metric space and show that int(A) is open, where int(A) is the set of all interior point of A.

   (b) If f is a mapping from a metric space (X,ρ) to a metric space (Y,d), show that f is continuous at a point $x_0 \in X$ if and only if for every sequence (x_n) in X converging to x_0, $f(x_n)$ converges to $f(x_0)$ in Y as $n\to\infty$.

6. Define a fixed point of a metric space (X,ρ) and show that every contraction mapping on a complete metric space has a unique fixed point.

MATH 303 2023/2024 SESSION

(1) Define a metric space and show that $(C[a,b], \rho)$ is a metric space where $C([a,b])$ is the space of all continuous real valued functions defined on $[a,b]$, and ρ is a function defined on $C([a,b])$ by
$$\rho(f,g) = \max_{t\in[a,b]} \{|f(t) - g(t)|\} \text{ with } f,g \in C([a,b])$$

Solution

Let X be a non-empty set. Then a metric d on X is a real valued function $d: X\times X \to \mathbb{R}$ that satisfies the following axioms.

(i) $d(x,y) \geq 0 \quad \forall x,y \in X$

(ii) $d(x,y) = 0 \quad \Leftrightarrow x = y \quad \forall x,y \in X$

(iii) $d(x,y) = d(y,x), \ \forall x,y \in X$

(iv) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in X$.

The pair (X,d) consisting of a non-empty set X and a metric d on it is called a metric space.

Verification of $[C([a,b]),d]$ as a metric space

m_1 $d(f,g) = \max_{t\in[a,b]} \{|f(t) - g(t)|\} \geq 0$ being the ~~supremum~~ maximum of non-negative numbers

m_2. $d(f,g) = \max_{t\in[a,b]} \{|f(t) - g(t)|\} = 0 \quad \Leftrightarrow |f(t) - g(t)| = 0$
$$\Leftrightarrow f(t) - g(t) = 0 \quad \forall t\in[a,b]$$
$$\Leftrightarrow f(t) = g(t) \quad \forall t\in[a,b]$$
$$\Leftrightarrow f = g$$

m_3 $d(f,g) = \max_{t\in[a,b]} \{|f(t) - g(t)|\} = \max_{t\in[a,b]} \{|g(t) - f(t)|\}$
$$= d(g,f).$$

m_4 $d(f,g) = \max_{t\in[a,b]} \{|f(t) - g(t)|\}$
$$= \max_{t\in[a,b]} \{|f(t) - h(t) + h(t) - g(t)|\}$$
$$\leq \max_{t\in[a,b]} \{|f(t) - h(t)|\} + \max_{t\in[a,b]} \{|h(t) - g(t)|\}$$
$$= d(f,h) + d(h,g) \ \square.$$

(1b) Show that (C, ρ) is a complete metric space, where C is the set of all convergent sequence of real numbers and ρ is defined as $\rho(x,y) = \sup\{|x_i - y_i| : i \in N\}$

<u>Verification</u>

M_1 $\rho(x,y) = \sup\{|x_i - y_i| : i \in N\} = 0$

$\Leftrightarrow |x_i - y_i| = 0 \quad \forall i \in N$

$\Leftrightarrow x_i - y_i = 0 \quad \forall i \in N$

$\Leftrightarrow x_i = y_i \quad \forall i \in N$

$\Leftrightarrow x = y$

m_2. $\rho(x,y) = \sup\{|x_i - y_i| : i \in N\} \geq 0$ being the supremum of non-negative numbers

m_3: $\rho(x,y) = \sup\{|x_i - y_i| : i \in N\}$

$= \sup\{|y_i - x_i| : i \in N\}$

$= \rho(y,x)$.

m_4 $\rho(x,y) = \sup\{|x_i - y_i| : i \in N\}$

$= \sup\{|x_i - z_i + z_i - y_i| : i \in N\}$

$\leq \sup\{|x_i - z_i| : i \in N\} + \sup\{|z_i - y_i| : i \in N\}$

$= \rho(x,z) + \rho(z,y)$. $\square$

(2) a. Define an open set in a metric space (X, ρ) and show that every open set is an open ball.

<u>Definition</u>: Let (X, ρ) be a metric space and $A \subseteq X$. Then A is said to be open in X if for all $a \in A$ $\exists r > 0 \ni B_r(a) \subseteq A$.

<u>Lemma</u>: Every open ball is an open set.

Let $B_r(x)$ be an arbitrary open ball in (X, ρ). We want to prove that $B_r(x)$ is an open set in X. So we must show that for every point $y \in B_r(x)$, we can find some $r_1 > 0$ $\ni B_{r_1}(y) \subseteq B_r(x)$.

Let y be arbitrary chosen in $B_r(x)$. Define $r_1 = r - \rho(x,y)$, since $y \in B_r(x)$, then $r - \rho(x,y) > 0$ $\Rightarrow r_1 > 0$. We assert that $B_{r_1}(y) \subseteq B_r(x)$. To prove this assertion, it suffices to show that every element in $B_{r_1}(y)$ is also in $B_r(x)$. Let $y_1 \in B_{r_1}(y)$ be arbitrary. Then we must show that $\rho(x,y_1) < r$.

Now $y_1 \in B_{r_1}(y) \Rightarrow \rho(y, y_1) < r$. Using triangle inequality we obtain

$\rho(x,y_1) \le \rho(x,y) + \rho(y,y_1)$
$< r_1 + \rho(x,y) = r - \rho(x,y) + \rho(x,y)$
$< r$

i.e $\rho(x,y) < r$ and $y_1 \in B_r(x)$. Thus $y \in B_{r_1}(y) \Rightarrow y_1 \in B_r(x)$

Thus, $B_{r_1}(y) \subseteq B_r(x)$

Since the point $y_1 \in B_r(x)$ is arbitrary, then the result follows. □.

(2b.) Show that an arbitrary union of open sets in (X,ρ) is open set in (X,ρ).

Proof

Let $\{B_\alpha : \alpha \in I\}$ where I is an arbitrary index set be a collection of open sets in (X,ρ). and Let $B = \bigcup_{\alpha \in I} B_\alpha$ be the arbitrary union of open sets in (X,ρ). Let $x \in B$ be arbitrary. But

$x \in B \left(= \bigcup_{\alpha \in I} B_\alpha\right) \Rightarrow x \in B_{\alpha_0}$ for some $\alpha_0 \in I$. Since B_{α_0} is open, then by definition $\exists\, r > 0 \ni B_r(x) \subseteq B_{\alpha_0} \subseteq \bigcup_{\alpha \in I} B_\alpha$

$\Rightarrow \forall x \in \bigcup_{\alpha \in I} B_\alpha, \exists\, r > 0 \ni B_r(x) \subseteq \bigcup_{\alpha \in I} B_\alpha$

Hence, $B = \bigcup_{\alpha \in I} B_\alpha$ is an open set in X. □

(3) Define a cluster point of a ~~set~~ subset F of a metric space (X,ρ) and show that arbitrary intersection of closed sets in closed.

Definition: Cluster point: Let F be any subset of an arbitrary metric space (X,d). Then, a point $x \in X$ (not necessarily in F) is said to be a cluster (limit) point of F if $\exists\, r > 0 \ni B_r(x) \cap F \neq \emptyset$.

(3b(i)) Arbitrary intersection of closed sets in closed.

Proof

Let $\{F_\alpha : \alpha \in I\}$ be the collection of closed sets in (X,ρ). Then we want to show that $\bigcap_{\alpha \in I} F_\alpha$ is closed. To see, this consider $\left(\bigcap_{\alpha \in I} F_\alpha\right)^c$. Since F_α is closed then each F_α is open.

$\left(\bigcap_{\alpha \in I} F_\alpha\right)^c = \bigcup_{\alpha \in I} F_\alpha^c$ (Demorgan's law)

But arbitrary union of open sets is open. So Let $x \in \left(\bigcap_{\alpha \in I} F_\alpha\right)^c$

Since $\bigcup_{\alpha \in I} F_\alpha^c$ is open, then $\exists\ r > 0 \ni B_r(x) \subseteq \bigcup_{\alpha \in I} F_\alpha^c$

$$\Rightarrow B_r(x) \subseteq \left(\bigcap_{\alpha \in I} F_\alpha\right)^c$$

Hence $\bigcap_{\alpha \in I} F_\alpha$ is closed.

(3b) show that $x \in \bar{F}$ if and only if there exists a sequence (x_n) in F such that (x_n) converges to x as $n \to \infty$.

Proof

To ~~show that~~ $\Rightarrow$ Let $x \in \bar{F}$ then by definition $\exists\ r > 0 \ni B_r(x) \cap F \neq \emptyset$.

Now in particular for $r = 1/n$, we see that $B_{1/n}(x) \cap F \neq \emptyset$.

i.e $\forall n \in \mathbb{N}\ \exists \{x_n\} \subseteq F \ni x_n \in B_{1/n}(x)$.

$\Rightarrow d(x_n, x) < 1/n \to 0$ as $n \to \infty$

$\Rightarrow x_n \to x$ as $n \to \infty$

Conversely, suppose that $\exists \{x_n\} \subseteq F$

$\ni x_n \to x$ as $n \to \infty$. Then by definition $\forall \epsilon > 0\ \exists\ n_0 \in \mathbb{N} \ni$

$d(x_n, x) < \epsilon\ \forall n \geq n_0$.

we get $d(x_{n_0}, x) < \epsilon$ and in particular for $\epsilon = r$, we get

$d(x_{n_0}, x) < r$

$\Rightarrow x_n \in B_r(x)$ and $B_r(x) \cap F \neq \emptyset$

Since $x_{n_0} \in F$. Thus $x \in \bar{F}$. □

4